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LETTER TO THE EDITOR

On the storage capacity of neural networks with sign-constrained weights

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Abstract. We derive the maximal storage capacity theorem for neural networks with sign-constrained weights using a simple geometrical argument and mention its implications for multi-layered networks with sign-constrained weights.

The properties of neural networks with sign-constrained weights have recently been investigated in detail. Both the dynamics [1, 2] and storage capacity [2, 3] of single-layer perceptron networks has been studied in the presence of sign constraints on the weights and learning rules have been considered for both single-layered and multi-layered networks with weight-sign constraints [4, 5]. For a set of weights W_{ij} the weight-sign constraints can be enforced by introducing a matrix g_{ij} with components ± 1 and performing learning in the presence of the constraints:

$$W_{ij} g_{ij} \geq 0 \,. \tag{1}$$

There are several motivations for considering neural networks with sign-constrained weights. They offer a mechanism for distinguishing recognition from non-recognition [6]. Also weight-sign constraints enforce a synaptic specificity (excitatory or inhibitory) at each synapse [7].

In the presence of weight-sign constraints the storage capacity is halved, this being independent of the weight-sign bias [3]. This interesting result was derived using the replica argument of Gardner and has since been extended by several authors [8–10]. In this letter we point out that this result emerges very straightforwardly from a geometrical argument based on a method originally formulated by Wendel [11], Cover [12] and others. Not only is this argument much simpler but it also gives a geometrical insight into the result.

Suppose we consider a single-layered network storing p random input vectors, ϕ_j^μ ($\mu = 1, .., p; j = 1, .., N$) each with a randomly assigned target value of 1 or -1 (at a particular output node, i). For a linearly separable problem these target values can be located as points (at coordinates $(\phi_1^\mu, \phi_2^\mu, ..., \phi_N^\mu)$) in an N-dimensional space and a learning rule is a procedure for finding a set of weights W_j such that $\sum_j W_j \phi_j^\mu > 0$ for every point with target output 1 and $\sum_j W_j \phi_j^\mu \leq 0$ for target -1. Thus the weight vector W_j is normal to a hyperplane (given by $\sum_j W_j \phi_j = 0$ for zero threshold) which dichotomizes or divides the space into points with target 1 and those with target -1.

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For neural networks with sign-constrained weights the weight vector normal to a hyperplane cannot assume any orientation but only those orientations compatible with the constraints in (1). Thus if a weight vector normal to a hyperplane is an allowed solution, the corresponding weight vector projecting in the opposite direction would not be allowed. In particular we note that a hyperplane induces one dichotomy of one point in N dimensions for this reason (e.g. if the target value for the pattern is 1 then it is not possible to obtain a target value of -1 for the same hyperplane without violating the sign constraints—this is in contrast to unconstrained weights where both orientations of the hyperplane are allowed).

For neural networks with sign-constrained weights storing random unbiased and uncorrelated pattern vectors, it is possible to count the number of ways in which a hyperplane can dichotomise p points in N dimensions, i.e. $C(p,N)$. We count hyperplanes which induce the same dichotomy as one and will call such sets of hyperplanes *equivalent*. As for networks with unconstrained weights [11, 12] it can be shown that this quantity satisfies the recurrence relation $C(p+1,N) = C(p,N) + C(p,N-1)$. To understand this relation we consider a space of p points in N dimensions (which we denote P) and consider the effect of adding an extra point (which we label x). Hyperplanes which induce a dichotomy of P are of two types: (i) they will only give one sign for the target value at x (we label these hyperplanes H_1); (ii) they may give either sign for the target value at x (we label these H_2). H_2 consists of a family of hyperplanes, equivalent in P, which can give either sign for the point at x and hence must include one hyperplane (labelled H_0) which includes the point x. If we take the vector ϕ_j^{p+1} (the coordinates of x) and project the points in P into a subspace normal to this vector then H_0 induces a dichotomy of this subspace. The $C(p,N-1)$ dichotomies of this subspace is equal to the number of dichotomies induced by the hyperplanes H_2 (since each set of P-equivalent hyperplanes has one H_0 dichotomizing this subspace). The recurrence relation then follows by observing that there are $C(p,N)$ dichotomies of P, and P-equivalent hyperplanes belonging to H_2 induce two dichotomies when x is included. Thus, if C_2 is the number of dichotomies induced by P-equivalent H_2 hyperplanes and C_1 the number induced by H_1 hyperplanes then $C(p+1,N) = 2C_2 + C_1$; $C(p,N) = C_1 + C_2$ and $C_2 = C(p,N-1)$ so $C(p+1,N) = C(p,N) + C(p,N-1)$.

This recurrence relation can be solved to give:

$$C(p,N) = \sum_{k=0}^{N-1} \binom{p-1}{k} C(1, N-k)\,. \tag{2}$$

Furthermore, as argued above, the orientation of the hyperplanes (imposed by (1)) gives $C(1,N) = 1$ for $N \geq 1$ and 0 otherwise.

Following Cover [12] we note that $C(p,N)/2^p$ is the probability of finding a weight set solution which gives the correct output for each input (since 2^p is the number of divisions of p points into two subsets). This probability passes through a well defined threshold [12] at the maximal storage capacity of the network. If we define this threshold as the condition that the probability of finding a solution is $\frac{1}{2}$, then from (2) we easily find that the maximal storage capacity is $p = N$. This is half the capacity of networks with unconstrained weights. Furthermore since $C(p,N)$ does not depend on the particular weight sign bias this maximal storage capacity is unaffected by the distribution of weight-signs imposed (i.e. whether the weights are all constrained to be positive, negative or a mixture of the two).

It is also possible to extend these arguments [13] to multi-layered networks with sign constraints for the weights in each layer [5]. As one may expect, compared with networks with unconstrained weights, the imposition of weight-sign constraints leads to a doubling of the lower bound on the number of nodes required in each layer to achieve an arbitrary dichotomy of p points.

References

[1] Wong K Y M and Campbell C 1991 The dynamics of neural networks with sign constrained weights *J. Phys. A: Math. Gen.* submitted

[2] Campbell C and Wong K Y M 1991 The dynamics and storage capacity of neural networks with sign constrained weights *Proc. Sitges 11 Conf. (Springer Lecture Notes in Physics)* (Berlin: Springer) to be published

[3] Amit D J, Campbell C and Wong K Y M 1989 *J. Phys. A: Math. Gen.* **22** 4687

[4] Amit D J, Wong K Y M and Campbell C 1989 *J. Phys. A: Math. Gen.* **22** 2039

[5] Campbell C, Karwatzki J M and Robinson A 1991 in preparation

[6] Shinomoto S 1987 *Biol. Cybern.* **57** 197

[7] Eccles J C 1964 *The Physiology of Synapses* (Berlin: Springer)

[8] Kurchan J and Domany E 1990 *J. Phys. A: Math. Gen.* **23** L847

[9] Kanter I and Eisenstein E 1990 *J. Phys. A: Math. Gen.* **23** L935

[10] Campbell C and Wong K Y M 1991 Neural networks with sign constrained weights storing correlated patterns, in preparation

[11] Wendel J G 1962 *Math. Scand.* **11** 109

[12] Cover T M 1965 *IEEE Trans. Electron. Comput.* **EC-14** 326

[13] Baum E B 1988 *J. Compl.* **4** 193